Problem Set 7

(Adapted from an old final exam)

It's OK to work together on problem sets.

**Questions 1, 2, and 3 are not as scary as they look. They are based on
the following model of public goods and the solution concepts of
competitive equilibrium, Lindahl equilibrium, and core:** Consider an
economy of a thousand (1000) identical households $i \in H$, a finite set of
firms F, and two commodities known as x and g. Each household i, is
endowed with (strictly positive) $\overline{X}^i$ of good x. Assume $\overline{X}^i > 1$. Good g is
produced by firms $j \in F$ (all of which have the same constant returns
technology), at the rate of one unit of output g for each unit of input x. g^i
denotes household i's purchase of good g.

$$G = \sum_{i \in H} g^i. \tag{1}$$

We define

Let each i have a continuous weakly concave utility function

$$u^i(x^i, G) \equiv x^i + (0.2)\min[G, 100] , \text{ for } x^i \text{ and } G \geq 0.$$

That is, household i enjoys G up to a maximum of 100 units and likes each
unit of G one fifth as much as he likes x. g^i and G are public goods. The
utility function is continuous everywhere, but it is not differentiable with
respect to G in the neighborhood of G=100. We contrast three solution
concepts below: competitive equilibrium, Lindahl equilibrium, and core.

Competitive Equilibrium
Assume marginal cost pricing: the price of x equals the price of g and we
can set these prices at unity, $p_x = 1 = p_g$, for convenience. All firms run zero
profits so household income is merely the value of endowment. We
maintain the convention that households sell all of endowment and
repurchase the amount they wish to consume.

Household i's budget constraint in a marginal cost pricing equilibrium
reads

$$x^i + g^i = \overline{X}^i \tag{2}$$

where x^i is i's purchase of good x, and g^i is i's purchase of good g (good x
acts as numeraire).
Household i's competitive market consumption choice problem is to

Choose x^i, g^i, to maximize $u^i(x^i, g^i + \sum_{h \neq i} g^h)$ subject to (2)   (3).

In solving (3), household i treats the prices of x and g parametrically and treats the choices of g^h of other households, h $\neq$ i, parametrically as well.

    We define a competitive equilibrium for this economy as choices x^{*i}, g^{*i}, $G^* = \sum_{h \in H} g^{*h}$, fulfilling (2) and (3) for each household i so that all markets clear, that is, so that

$$G^* + \sum_{h \in H} x^{*h} = \sum_{h \in H} \bar{X}^h \qquad (4).$$

Lindahl Equilibrium

    We define the Lindahl budget constraint of household i as

$$x^i + q^i G^i = \bar{X}^i \qquad (5)$$

where q^i is i's (personal) Lindahl price of the public good. Household i's Lindahl consumption choice problem is to

    Choose x^i, G^i to maximize $u^i(x^i, G^i)$ subject to (5)   (6)

where i treats q^i parametrically. We define a Lindahl equilibrium of the economy as an array of choices x^{*i}, G^{*i}, prices q^i, i$\in$H, fulfilling (6), so that $\sum_{i \in H} q^i = 1$, so that all G^{*i} (=G^*), i$\in$H, are equal and markets clear, that is, (4) is fulfilled.

Core

    An allocation G, x^i, i$\in$H, is said to be "attainable" if (4) holds, that is, if

$$G + \sum_{i \in H} x^i = \sum_{i \in H} \bar{X}^i \qquad (7).$$

An attainable allocation x^i (i$\in$H), G, is be said to be 'blocked' if there is a coalition, S $\subset$ H, so that we can find y^i, for i $\in$ S, so that $u^i(y^i, G^S) \geq u^i(x^i, G)$ for all i$\in$S with a strict inequality for some i$\in$S, so that $\sum_{i \in S} y^i \leq \sum_{i \in S} \bar{X}^i$, where we define $G^S = G - \sum_{i \in S}(y^i - x^i)$. The term $-\sum_{i \in S}(y^i - x^i)$ is the change in the provision of g by members of S compared to the blocked allocation x^i, G. That is, an allocation is blocked when a coalition believes that it can advantageously reallocate its endowment (while assuming that the complementary coalition's provision of good g, g^h for h$\in$H\S, remains unchanged). An allocation is said to be in the 'core' if it is attainable and unblocked.

1. There is a competitive equilibrium in this problem.
(i) Find competitive equilibrium prices and the resulting allocation.
Explain.
(ii) The First Fundamental Theorem of Welfare Economics (Starr's *General Equilibrium Theory: An Introduction* , Theorem 12.1) says that the competitive equilibrium allocation is Pareto efficient. Is that true in this example? Why or why not?
(iii) Provision of public goods is thought to generate a 'free rider' problem. Is that true in this example? Explain.

2. (i) Find Lindahl equilibrium prices and the corresponding Lindahl equilibrium allocation. This is probably simplest if you choose a Lindahl equilibrium that treats all households equally.
(ii) Is the Lindahl equilibrium allocation Pareto efficient? Explain.
(iii) Provision of public goods is thought to generate a 'free rider' problem. Is that true in this example? Explain.

3. (i) Demonstrate that the core of this economy is empty. That is, there is no allocation (x^i, g^i) for $i \in H$, that is unblocked. You may assume an equal treatment property: at a core allocation, all households with identical tastes and endowment achieve identical utility.
Hint: A core allocation is always Pareto efficient since a Pareto inefficient allocation would be blocked by $S = H$. Find the equal-treatment Pareto efficient allocation. It is unique. Show that it is blocked by any singleton (coalition consisting of a single household).
(ii) Provision of public goods is thought to generate a 'free rider' problem. Is that true in this example? Explain.

4. (Adapted from Walter P. Heller with thanks to David Starrett)
Consider a Robinson Crusoe economy producing two outputs on a river. The upstream firm produces commodity x and water pollution in proportion to the output of commodity x. The downstream firm produces y, and output of y is reduced by the pollution coming from production of x. There is a lower bound of 0 on the output of x and y. The production relations of these commodities are

$$x = L_x$$
$$y = L_y - x \text{ when this expression} > 0$$

0 otherwise.

where L_x, L_y is the amount of labor going to production of x and y respectively. L_x, $L_y \geq 0$. Labor is inelastically supplied and leisure is not valued. $L_x + L_y = 2000$. Note that the production possibility set is nonconvex.

Robinson's utility function is

$u(x, y) = 6x + 4y.$

His income is precisely sufficient to purchase all of the goods x and y produced.

   (i) The downstream firm treats the volume of x upstream parametrically. A Pigouvian tax on good x, τ, that will correctly reflect the external effect will have the property $\tau = p_y$. Then the price paid by buyers of x is $p_x + \tau$ but the price received by sellers is p_x.

   Show that the allocation

   $x = 0, \ y = 2000 ,$

is a competitive equilibrium with taxation, where $p_x = p_y = 1 = \tau$. You may assume the wage rate is 1. This is a corner solution so the first order conditions may be fulfilled as an inequality.

   (ii) The allocation $x = 0, \ y = 2000$, fulfills the first order conditions for a local maximum of utility subject to technology constraint and hence for a Pareto efficient allocation. That is, $MRT_{x,y} = 2 > 1.5 = MRS_{x,y}$; the inequality is appropriate at a corner solution. In a convex economy, the first order conditions would be sufficient for Pareto efficiency, but this economy is nonconvex. Show that the allocation is Pareto inefficient. (Hint: It may help to diagram this problem).

   (iii) Assume a Cobb-Douglas utility function: $v(x,y) = x^{(1/2)}y^{(1//2)}$. Solve for a Pareto efficient allocation. Can the allocation be supported as a competitive equilibrium with Pigouvian taxation?